Statistical mechanics of lossy data compression using a non-monotonic perceptron

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The performance of a lossy data compression scheme for uniformly biased Boolean messages is investigated via methods of statistical mechanics. Inspired by a formal similarity to the storage capacity problem in neural network research, we utilize a perceptron of which the transfer function is appropriately designed in order to compress and decode the messages. Employing the replica method, we analytically show that our scheme can achieve the optimal performance known in the framework of lossy compression in most cases when the code length becomes infinite. The validity of the obtained results is numerically confirmed.

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I. INTRODUCTION

Recent active research on error-correcting codes (ECC) has revealed a great similarity between information theory (IT) and statistical mechanics (SM) [1, 2, 3, 4, 5, 6, 7]. As some of these studies have shown that methods from SM can be useful in IT, it is natural to expect that a similar approach may also bring about novel developments in fields other than ECC.

The purpose of the present paper is to offer such an example. More specifically, we herein employ methods from SM to analyze and develop a scheme of data compression. Data compression is generally classified into two categories; lossless and lossy compression [8]. The purpose of lossless compression is to reduce the size of messages in information representation under the constraint of perfect retrieval. The message length in the framework of lossy compression can be further reduced by allowing a certain amount of distortion when the original expression is retrieved.

The possibility of lossless compression was first pointed out by Shannon in 1948 in the *source coding theorem* [9], whereas the counterpart of lossy compression, termed the *rate-distortion theorem*, was presented in another paper by Shannon more than ten years later [10]. Both of these theorems provide the best possible compression performance in each framework. However, their proofs are not constructive and suggest few clues for how to design practical codes. After much effort had been made for achieving the optimal performance in practical time scales, a practical lossless compression code that asymptotically saturates the source-coding limit was discovered [11]. Nevertheless, thus far, regarding lossy compression, no algorithm which can be performed in a practical time scale saturating the optimal performance predicted by the rate-distortion theory has been found, even for simple information sources. Therefore, the quest for better lossy compression codes remains one of the important problems in IT [8, 12, 13, 14].

Therefore, we focus on designing an efficient lossy compression code for a simple information source of uniformly biased Boolean sequences. Constructing a scheme of data compression requires implementation of a map from compressed data of which the redundancy should be minimized, to the original message which is somewhat biased and, therefore, seems redundant. However, since the summation over the Boolean field generally reduces the statistical bias of the data, constructing such a map for the aforementioned purpose by only linear operations is difficult, although the best performance can be achieved by such linear maps in the case of ECC [1, 2, 4, 5, 7] and lossless compression [15]. In contrast, producing a biased output from an unbiased input is relatively easy when a non-linear map is used. Therefore, we will employ a perceptron of which the transfer function is optimally designed in order to devise a lossy compression scheme.

The present paper is organized as follows. In the next section, we briefly introduce the framework of lossy data compression, providing the optimal compression performance which is often expressed as the *rate-distortion function* in the case of the uniformly biased Boolean sequences. In section III, we explain how to employ a non-monotonic perceptron to compress and decode a given message. The ability and limitations of the proposed scheme are examined using the replica method in section IV. Due to a specific (mirror) symmetry that we impose on the transfer function of the perceptron, one can *analytically* show that the proposed method can saturate the rate-distortion function for most choices of parameters when the code length becomes infinite. The

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obtained results are numerically validated by means of the extrapolation on data from systems of finite size in section V. The final section is devoted to summary and discussion.

II. LOSSY DATA COMPRESSION

Let us first provide the framework of lossy data compression. In a general scenario, a redundant original message of M random variables $\boldsymbol{y} = (y^1, y^2, \ldots, y^M)$, which we assume here as a Boolean sequence $y^\mu \in \{0,1\}$, is compressed into a shorter (Boolean) expression $\boldsymbol{s} = (s_1, s_2, \ldots, s_N)$ $(s_i \in \{0,1\}, N < M)$. In the decoding phase, the compressed expression $\boldsymbol{s}$ is mapped to a representative message $\tilde{\boldsymbol{y}} = (\tilde{y}^1, \tilde{y}^2, \ldots, \tilde{y}^M)$ $(\tilde{y}^\mu \in \{0,1\})$ in order to retrieve the original expression (Fig. 1).

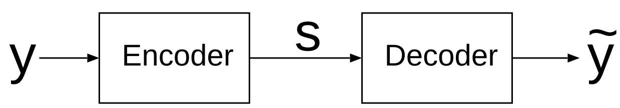


FIG. 1: Encoder and decoder in the framework of lossy compression. The retrieved sequence $\tilde{\boldsymbol{y}}$ need not be identical to the original sequence $\boldsymbol{y}$.

In the source coding theorem, it is shown that perfect retrieval $\tilde{\boldsymbol{y}} = \boldsymbol{y}$ is possible if the compression rate $R = N/M$ is greater than the entropy per bit of the message $\boldsymbol{y}$ when the message lengths M and N become infinite. On the other hand, in the framework of lossy data compression, the achievable compression rate can be further reduced allowing a certain amount of distortion between the original and representative messages $\boldsymbol{y}$ and $\tilde{\boldsymbol{y}}$.

A measure to evaluate the distortion is termed the *distortion function*, which is denoted as $d(\boldsymbol{y}, \tilde{\boldsymbol{y}}) \geq 0$. Here, we employ the Hamming distance

$$d(\boldsymbol{y}, \tilde{\boldsymbol{y}}) = \sum_{\mu=1}^{M} d(y^\mu, \tilde{y}^\mu), \qquad (1)$$

where

$$d(y^\mu, \tilde{y}^\mu) = \begin{cases} 0 & \text{if} \quad y^\mu = \tilde{y}^\mu \\ 1 & \text{if} \quad y^\mu \neq \tilde{y}^\mu \end{cases}, \qquad (2)$$

as is frequently used for Boolean messages.

Since the original message $\boldsymbol{y}$ is assumed to be generated randomly, it is natural to evaluate the average of Eq. (1). This can be performed by averaging $d(\boldsymbol{y}, \tilde{\boldsymbol{y}})$ with respect to the joint probability of $\boldsymbol{y}$ and $\tilde{\boldsymbol{y}}$ as

$$\overline{d(\boldsymbol{y}, \tilde{\boldsymbol{y}})} = \sum_{\boldsymbol{y}} \sum_{\tilde{\boldsymbol{y}}} P(\boldsymbol{y}, \tilde{\boldsymbol{y}}) d(\boldsymbol{y}, \tilde{\boldsymbol{y}}). \qquad (3)$$

By allowing the average distortion per bit $\overline{d(\boldsymbol{y}, \tilde{\boldsymbol{y}})}/M$ up to a given permissible error level $0 \leq D \leq 1$, the achievable compression rate can be reduced below the entropy per bit. This limit $R(D)$ is termed the *rate-distortion function*, which provides the optimal compression performance in the framework of lossy compression.

The rate-distortion function is formally obtained as a solution of a minimization problem with respect to the mutual information between $\boldsymbol{y}$ and $\tilde{\boldsymbol{y}}$ [8]. Unfortunately, solving the problem is generally difficult and analytical expressions of $R(D)$ are not known in most cases.

The uniformly biased Boolean message in which each component is generated independently from an identical distribution $P(y^\mu = 1) = 1 - P(y^\mu = 0) = p$ is one of the exceptional models for which $R(D)$ can be analytically obtained. For this simple source, the rate-distortion function becomes

$$R(D) = H_2(p) - H_2(D), \qquad (4)$$

where $H_2(x) = -x \log_2 x - (1-x) \log_2 (1-x)$.

However, it should be addressed here that a practical code that saturates this limit has not yet been reported, even for this simplest model. Therefore, in the following, we focus on this information source and look for a code that saturates Eq. (4) examining properties required for good compression performance.

III. COMPRESSION BY PERCEPTRON

In a good compression code for the uniformly biased source, it is conjectured that compressed expressions $\boldsymbol{s}$ should have the following properties:

(I) In order to minimize loss of information in the original expressions, the entropy per bit in $\boldsymbol{s}$ must be maximized. This implies that the components of $\boldsymbol{s}$ are preferably unbiased and uncorrelated.

(II) In order to reduce the distortion, the representative message $\tilde{\boldsymbol{y}}(\boldsymbol{s})$ should be placed close to the typical sequences of the original messages which are biased.

Unfortunately, it is difficult to construct a code that satisfies both of the above two requirements utilizing only linear transformations over the Boolean field while such maps provide the optimal performance in the case of ECC [1, 2, 4, 5, 7] and lossless compression [15]. This is because a linear transformation generally reduces statistical bias in messages, which implies that the second requirement **(II)** cannot be realized for unbiased and uncorrelated compressed expressions $\boldsymbol{s}$ that are preferred in the first requirement **(I)**.

One possible method to design a code that has the above properties is to introduce a non-linear transformation. A perceptron provides one of the simplest schemes for carrying out this task.

In order to simplify notations, let us replace all the Boolean expressions $\{0,1\}$ with binary ones $\{1,-1\}$. By

this, we can construct a non-linear map from the compressed message $\boldsymbol{s}$ to the retrieved sequence $\tilde{\boldsymbol{y}}$ utilizing a perceptron as

$$\tilde{y}^{\mu} = f\left(\frac{1}{\sqrt{N}}\boldsymbol{s}\cdot\boldsymbol{x}^{\mu}\right) \quad (\mu = 1, 2, \ldots, M), \tag{5}$$

where $\boldsymbol{x}^{\mu=1,2,\ldots,M}$ are fixed N-dimensional vectors to specify the map and $f(\cdot)$ is a transfer function from a real number to a binary variable $\tilde{y}^{\mu} \in \{1, -1\}$ that should be optimally designed.

Since each component of the original message $\boldsymbol{y}$ is produced independently, it is preferred to minimize the correlations among components of a representative vector $\tilde{\boldsymbol{y}}$, which intuitively indicates that random selection of $\boldsymbol{x}^{\mu}$ may provide a good performance. Therefore, we hereafter assume that vectors $\boldsymbol{x}^{\mu=1,2,\ldots,M}$ are independently drawn from the N-dimensional normal distribution $P(\boldsymbol{x}) = (2\pi)^{-N/2}\exp\left[-|\boldsymbol{x}|^2/2\right]$.

Based on the non-linear map (5), a lossy compression scheme can be defined as follows:

- **Compression:** For a given message $\boldsymbol{y}$, find a vector $\boldsymbol{s}$ that minimizes the distortion $d(\boldsymbol{y}, \tilde{\boldsymbol{y}}(\boldsymbol{s}))$, where $\tilde{\boldsymbol{y}}(\boldsymbol{s})$ the representative vector which is generated from $\boldsymbol{s}$ by Eq. (5). The obtained $\boldsymbol{s}$ is the compressed message.

- **Decoding:** Given the compressed message $\boldsymbol{s}$, the representative vector $\tilde{\boldsymbol{y}}(\boldsymbol{s})$ produced by Eq. (5) provides the approximate message for the original message.

Here, we should notice that the formulation of the current problem has become somewhat similar to that for the storage capacity evaluation of the Ising perceptron [16, 17] regarding $\boldsymbol{s}$, $\boldsymbol{x}^{\mu}$ and y^{μ} as "Ising couplings", "random input pattern" and "random output", respectively. Actually, the rate-distortion limit in the current framework for $D = 0$ and $p = 1/2$ can be calculated as the inverse of the storage capacity of the Ising perceptron, α_c^{-1}.

This observation implies that the simplest choice of the transfer function $f(u) = \mathrm{sign}(u)$, where $\mathrm{sign}(u) = 1$ for $u \geq 0$ and -1 otherwise, does not saturate the rate-distortion function (4). This is because the well-known storage capacity of the simple Ising perceptron, $\alpha_c = M/N \approx 0.83$, means that the "compression limit" achievable by this monotonic transfer function becomes $R_c = N/M = \alpha_c^{-1} \approx 1.20$ and far from the value provided by Eq. (4) for this parameter choice $R(D = 0) = H_2(p = 1/2) - H_2(D = 0) = 1$. We also examined the performances obtained by the monotonic transfer function for biased messages $0 < p < 1/2$ by introducing an adaptive threshold in our previous study [18] and found that the discrepancy from the rate-distortion function becomes large in particular for relatively high R while fairly good performance is observed for low rate regions.

Therefore, we have to design a non-trivial function $f(\cdot)$ in order to achieve the rate-distortion limit, which may seem hopeless as there are infinitely many degrees of freedom to be tuned. However, a useful clue exists in the literature of perceptrons, which have been investigated extensively during the last decade.

In the study of neural network, it is widely known that employing a non-monotonic transfer function can highly increase the storage capacity of perceptrons [19]. In particular, Bex *et al.* reported that the capacity of the Ising perceptron that has a transfer function of the reversed-wedge type $f(u) = f_{\mathrm{RW}}(u) = \mathrm{sign}(u-k)\mathrm{sign}(u)\mathrm{sign}(u+k)$ can be maximized to $\alpha_c = 1$ by setting $k = \sqrt{2\ln 2}$ [20], which implies that the rate-distortion limit $R = 1$ is achieved for the case of $p = 1/2$ and $D = 0$ in the current context. Although not explicitly pointed out in their paper, the most significant feature observed for this parameter choice is that the Edwards-Anderson (EA) order parameter $(1/N)\left|\langle\boldsymbol{s}\rangle\right|^2$ vanishes to zero, where $\langle\cdots\rangle$ denotes the average over the posterior distribution given $\boldsymbol{y}$ and $\boldsymbol{x}^{\mu=1,2,\ldots,M}$. This implies that the dynamical variable $\boldsymbol{s}$ in the posterior distribution given $\boldsymbol{y}$ and $\boldsymbol{x}^{\mu=1,2,\ldots,M}$ is unbiased and, therefore, the entropy is maximized, which meets the first requirement **(I)** addressed above. Thus, designing a transfer function $f(u)$ so as to make the EA order parameter vanish seems promising as the first discipline for constructing a good compression code.

However, the reversed-wedge type transfer function $f_{\mathrm{RW}}(u)$ is not fully satisfactory for the present purpose. This is because this function cannot produce a biased sequence due to the symmetry $f_{\mathrm{RW}}(-u) = -f_{\mathrm{RW}}(u)$, which means that the second requirement **(II)** provided above would not be satisfied for $p \neq 0.5$.

Hence, another candidate for which the EA parameter vanishes and the bias of the output can be easily controlled must be found. A function that provides these properties was once introduced for reducing noise in signal processing, such as $f_{\mathrm{LA}}(u) = \mathrm{sign}\left(k - |u|\right)$ [21, 22] (Fig. 2). Since this locally activated function has mirror symmetry $f_{\mathrm{LA}}(-u) = f_{\mathrm{LA}}(u)$, both $\boldsymbol{s}$ and $-\boldsymbol{s}$ provide identical output for any input, which means that the EA parameter is likely to be zero. Moreover, one can easily control the bias of output sequences by adjusting the value of the threshold parameter k. Therefore, this transfer function looks highly promising as a useful building-block for constructing a good compression code.

In the following two sections, we examine the validity of the above speculation, analytically and numerically evaluating the performance obtained by the locally activated transfer function $f_{\mathrm{LA}}(u)$.

IV. ANALYTICAL EVALUATION

We here analytically evaluate the typical performance of the proposed compression scheme using the replica method. Our goal is to calculate the minimum permissible average distortion D when the compression rate $R = N/M$ is fixed. The analysis is similar to that of the storage capacity for perceptrons.

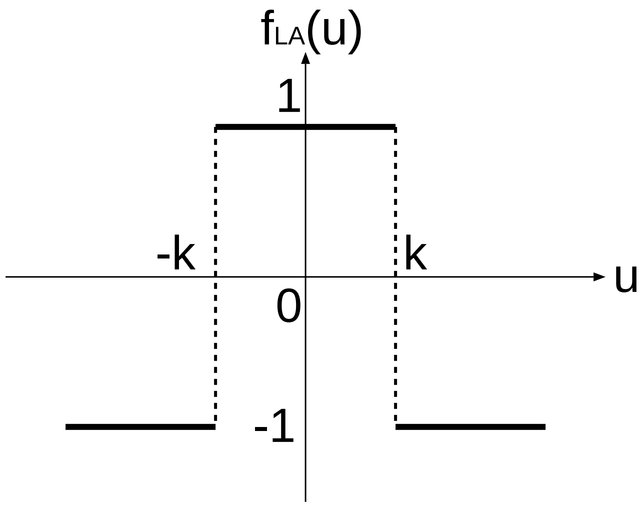


FIG. 2: Input-output relation of $f_{\rm LA}(u)$.

Employing the Ising spin expression, the Hamming distortion can be represented as

$$d(\boldsymbol{y},\tilde{\boldsymbol{y}}(\boldsymbol{s}))=\sum_{\mu=1}^{M}\left\{1-\Theta_k(u^\mu;y^\mu)\right\}, \qquad (6)$$

where

$$\Theta_k(u;1) \;=\; 1-\Theta_k(u;-1)=\begin{cases}1, & \text{for } |u|\le k\\ 0, & \text{otherwise}\end{cases}, \qquad (7)$$

$$u^\mu \;=\; \frac{1}{\sqrt{N}}\boldsymbol{s}\cdot\boldsymbol{x}^\mu. \qquad (8)$$

Then, for a given original message $\boldsymbol{y}$ and vectors $\boldsymbol{x}^{\mu(=1,2,\ldots,M)}$, the number of dynamical variables $\boldsymbol{s}$ which provide a fixed Hamming distortion $d(\boldsymbol{y},\tilde{\boldsymbol{y}}(\boldsymbol{s}))=MD$ $(0\le D\le 1)$, can be expressed as

$$\mathcal{N}(D)=\mathop{\rm Tr}_{\boldsymbol{s}}\,\delta(MD-d(\boldsymbol{y},\tilde{\boldsymbol{y}}(\boldsymbol{s}))). \qquad (9)$$

Since $\boldsymbol{y}$ and $\boldsymbol{x}^\mu$ are randomly generated predetermined variables, the quenched average of the entropy per bit over these parameters

$$S(D)=\frac{\langle \ln\mathcal{N}(D)\rangle_{\boldsymbol{y},\boldsymbol{x}}}{N}, \qquad (10)$$

to which the raw entropy per bit $(1/N)\ln\mathcal{N}(D)$ becomes identical for most realizations of $\boldsymbol{y}$ and $\boldsymbol{x}^\mu$, is naturally introduced for investigating the typical properties. This can be performed by the replica method $(1/N)\left\langle\ln\mathcal{N}(D)\right\rangle_{\boldsymbol{y},\boldsymbol{x}} = \lim_{n\to 0}(1/nN)\ln\left\langle\mathcal{N}^n(D)\right\rangle_{\boldsymbol{y},\boldsymbol{x}}$, analytically continuing the expressions of $\left\langle\mathcal{N}^n(D)\right\rangle_{\boldsymbol{y},\boldsymbol{x}}$ obtained for natural numbers n to non-negative real number n [23, 24].

When n is a natural number, $\mathcal{N}^n(D)$ can be expanded to a summation over n-replicated systems as $\mathcal{N}^n(D)=\mathrm{Tr}_{\boldsymbol{s}^1,\boldsymbol{s}^2,\ldots,\boldsymbol{s}^n}\prod_{a=1}^n\delta(MD-d(\boldsymbol{y},\tilde{\boldsymbol{y}}(\boldsymbol{s}^a)))$, where the subscript a denotes a replica index. Inserting an identity

$$\begin{aligned}1 \;&=\; \prod_{a>b}\int_{-\infty}^{+\infty}dq_{ab}\delta\left(\boldsymbol{s}^a\cdot\boldsymbol{s}^b-Nq_{ab}\right)\\ &=\left(\frac{1}{2\pi i}\right)^{n(n-1)/2}\int_{-\infty}^{+\infty}\prod_{a>b}dq_{ab}\int_{-i\infty}^{+i\infty}\prod_{a>b}d\hat{q}_{ab}\\ &\quad\exp\left[\sum_{a>b}\hat{q}_{ab}\left(\boldsymbol{s}^a\cdot\boldsymbol{s}^b-Nq_{ab}\right)\right] \end{aligned} \qquad (11)$$

into this expression and utilizing the Fourier expression of the delta function

$$\begin{aligned}&\delta(MD-d(\boldsymbol{y},\tilde{\boldsymbol{y}}(\boldsymbol{s}^a)))\\ &=\int_{-i\infty}^{+i\infty}\frac{d\beta_a}{2\pi i}\exp\left[\beta_a(MD-d(\boldsymbol{y},\tilde{\boldsymbol{y}}(\boldsymbol{s}^a)))\right],\end{aligned} \qquad (12)$$

we can calculate the moment $\left\langle\mathcal{N}^n(D)\right\rangle_{\boldsymbol{y},\boldsymbol{x}}$ for natural numbers $n=1,2,3,\ldots$ as

$$\begin{aligned}\left\langle\mathcal{N}^n(D)\right\rangle_{\boldsymbol{y},\boldsymbol{x}} \;\sim\; &\int\prod_a d\beta_a\int\prod_{a>b}dq_{ab}\int\prod_{a>b}d\hat{q}_{ab}\\ &\exp N\left[R^{-1}\ln\left\langle\int d\boldsymbol{v}\;d\boldsymbol{u}\exp\left(-\frac{1}{2}\boldsymbol{v}^tQ\boldsymbol{v}+i\boldsymbol{v}\cdot\boldsymbol{u}\right)\prod_{a=1}^n\left\{e^{-\beta_a}+(1-e^{-\beta_a})\Theta_k\left(u_a;y\right)\right\}\right\rangle_y\right.\\ &\left.\qquad+\ln\left\{\mathop{\rm Tr}_{\{s^a\}}\;\exp\left(\sum_{a>b}\hat{q}_{ab}s^as^b\right)\right\}-\sum_{a>b}q_{ab}\hat{q}_{ab}+R^{-1}D\sum_{a=1}^n\beta_a\right],\end{aligned} \qquad (13)$$

where Q is an $n\times n$ matrix of which elements are given by the parameters $\{q_{ab}\}$ and $\langle\cdots\rangle_y$ =

$\sum_{y=\pm 1}(p\delta(y-1)+(1-p)\delta(y+1))(\cdots)$.

In the thermodynamic limit $N, M \to \infty$ keeping the compression rate R finite, this integral can be evaluated via a saddle point problem with respect to macroscopic variables q_{ab}, $\hat{q}_{ab}$ and β_a.

In order to proceed further, a certain ansatz about the symmetry of the replica indices must be assumed. We here assume the simplest one, that is, the replica symmetric (RS) ansatz

$$\beta_a = \beta, \quad q_{ab} = q, \quad \hat{q}_{ab} = \hat{q} \qquad (\forall a > b), \tag{14}$$

for which the saddle point expression of Eq. (13) is likely to hold for any real number n. Taking the limit $n \to 0$ of this expression, we obtain

$$\begin{aligned} S(D) &= \lim_{n\to 0} \frac{\ln \langle \mathcal{N}^n(D)\rangle_{\boldsymbol{y},\boldsymbol{x}}}{Nn} \\ &= \operatorname*{extr}_{\beta,q,\hat{q}} \left\{ R^{-1}\left[p\int Dt\ \ln\left\{e^{-\beta}+(1-e^{-\beta})\left\{H(w_1)-H(w_2)\right\}\right\} \right.\right. \\ &\qquad \left. +(1-p)\int Dt\ \ln\left\{e^{-\beta}+(1-e^{-\beta})\left\{-H(w_1)+H(w_2)+1\right\}\right\}\right] \\ &\qquad \left. -\frac{\hat{q}(1-q)}{2} + \int Du\ \left\{\ln(2\cosh\sqrt{\hat{q}}u)\right\} + R^{-1}\beta D \right\}, \end{aligned} \tag{15}$$

where $w_1 = \frac{-k-\sqrt{q}t}{\sqrt{1-q}}$, $w_2 = \frac{k-\sqrt{q}t}{\sqrt{1-q}}$, $Dx = \frac{dx}{\sqrt{2\pi}}\exp(-x^2/2)$ and $H(x) = \int_x^\infty Dt$. $\operatorname{extr}\{\cdots\}$ denotes the extremization. Under this RS ansatz, the macroscopic variable q indicates the EA order parameter as $q = (1/N)\left|\langle \boldsymbol{s}\rangle\right|^2$. The validity of this solution will be examined later.

Since the dynamical variable $\boldsymbol{s}$ is discrete in the current system, the entropy (15) must be non-negative. This indicates that the achievable limit for a fixed compression rate R and a transfer function $f_{\rm LA}(u)$ which is specified by the threshold parameter k can be characterized by a transition depicted in Fig. 3.

Utilizing the Legendre transformation $\beta F(\beta) = \min_D\{R^{-1}\beta D - S(D)\}$, the *free energy* $F(\beta)$ for a fixed *inverse temperature* β, which is an external parameter and should be generally distinguished from the variational variable β in Eq. (15), can be derived from $S(D)$. This implies that the distortion $D(\beta)$ that minimizes $R^{-1}\beta D - S(D)$ and of which the value is computed from $F(\beta)$ as $D(\beta) = \partial(\beta F(\beta))/\partial(R^{-1}\beta)$ can be achieved by randomly drawing $\boldsymbol{s}$ from the canonical distribution $P(\boldsymbol{s}|\boldsymbol{y},\boldsymbol{x}^\mu) \sim \exp[-\beta d(\boldsymbol{y},\tilde{\boldsymbol{y}}(\boldsymbol{s}))]$ which is provided by the given β. For a modest β, the achieved distortion $D(\beta)$ is determined as a point for which the slope of $S(D)$ becomes identical to $R^{-1}\beta$ and $S(D) > 0$ (Fig. 3 (a)). As β becomes higher, $D(\beta)$ moves to the left, which indicates that the distortion can be reduced by introducing a lower temperature. However, at a critical value β_c characterized by the condition $S(D(\beta_c)) = 0$ (Fig. 3 (b)), the number of states that achieve $D(\beta_c)$ which is the typical value of $\min_{\boldsymbol{s}}\{d(\boldsymbol{y},\tilde{\boldsymbol{y}}(\boldsymbol{s}))\}$ vanishes to zero. Therefore, for $\beta > \beta_c$, $D(\beta)$ is fixed to $D(\beta_c)$ and the distortion $D < D(\beta_c)$ is not achievable (Fig. 3 (c)).

The above argument indicates that the limit of the achievable distortion $D(\beta_c)$ for a given rate R and a threshold parameter k in the current scheme can be evaluated from conditions

$$D(\beta) = \frac{\partial(\beta F(\beta))}{\partial(R^{-1}\beta)}, \tag{16}$$

$$S(D(\beta)) = 0, \tag{17}$$

being parameterized by the inverse temperature β.

Due to the mirror symmetry $f_{\rm LA}(-u) = f_{\rm LA}(u)$, $q = \hat{q} = 0$ becomes the saddle point solution for the extremization problem (15) as we speculated in the previous section, and no other solution is discovered. Inserting $q = \hat{q} = 0$ into the right-hand side of Eq. (15) and employing the Legendre transformation, the free energy is obtained as

$$\begin{aligned} \beta F(\beta) &= -\ln 2 - R^{-1}\left[p\ln\left\{e^{-\beta}+(1-e^{-\beta})A_k\right\}\right. \\ &\qquad \left. +(1-p)\ln\left\{e^{-\beta}+(1-e^{-\beta})(1-A_k)\right\}\right], \end{aligned} \tag{18}$$

where $A_k = 1-2H(k)$, which means that Eqs. (16) and (17) yield

$$\begin{aligned} D &= p\frac{e^{-\beta}-e^{-\beta}A_k}{e^{-\beta}+(1-e^{-\beta})A_k} \\ &+ (1-p)\frac{e^{-\beta}-e^{-\beta}(1-A_k)}{e^{-\beta}+(1-e^{-\beta})(1-A_k)}, \end{aligned} \tag{19}$$

and

$$\begin{aligned} R &= -\left[p\log_2\left\{e^{-\beta}+(1-e^{-\beta})A_k\right\}\right. \\ &+ \left.(1-p)\log_2\left\{e^{-\beta}+(1-e^{-\beta})(1-A_k)\right\}\right] \\ &- \frac{\beta}{\ln 2}\left[p\frac{e^{-\beta}-e^{-\beta}A_k}{e^{-\beta}+(1-e^{-\beta})A_k}\right. \\ &+ \left.(1-p)\frac{e^{-\beta}-e^{-\beta}(1-A_k)}{e^{-\beta}+(1-e^{-\beta})(1-A_k)}\right], \end{aligned} \tag{20}$$

respectively.

The rate-distortion function $R(D)$ represents the optimal performance that can be achieved by appropriately tuning the scheme of compression. This means that $R(D)$ can be evaluated as the convex hull of a region in the D-R plane defined by Eqs. (19) and (20) by varying the inverse temperature β and the threshold parameter k (or A_k). Minimizing R for a fixed D, one can show that the relations

$$e^{-\beta} = \frac{D}{1-D}, \tag{21}$$

$$e^{-\beta} + (1-e^{-\beta})A_k = \frac{p}{1-D}, \tag{22}$$

are satisfied at the convex hull, which offers the optimal choice of parameters β and k as functions of a given permissible distortion D and a bias p. Plugging these into Eq. (20), we obtain

$$\begin{aligned} R &= R_{\mathrm{RS}}(D) = -p\log_2 p - (1-p)\log_2(1-p) \\ &\quad +D\log_2 D + (1-D)\log_2(1-D) \\ &= H_2(p) - H_2(D), \end{aligned} \tag{23}$$

which is identical to the rate-distortion function for uniformly biased binary sources (4).

The results obtained thus far indicate that the proposed scheme achieves the rate-distortion limit when the threshold parameter k is optimally adjusted. However, since the calculation is based on the RS ansatz, we must confirm the validity of assuming this specific solution. We therefore examined two possible scenarios for the breakdown of the RS solution.

The first scenario is that the local stability against the fluctuations for disturbing the replica symmetry is broken, which is often termed the Almeida-Thouless (AT) instability [25], and can be examined by evaluating the excitation of the free energy around the RS solution. As the current RS solution can be simply expressed as $q = \hat{q} = 0$, the condition for this solution to be stable can be analytically obtained as

$$R > R_{\mathrm{AT}}(D) = \frac{1}{p(1-p)}\left\{\frac{2k(1-2D)}{\sqrt{2\pi}}e^{-\frac{k^2}{2}}\right\}^2. \tag{24}$$

In most cases, the RS solution satisfies the above condition and, therefore, does not exhibit the AT instability. However, we found numerically that for relatively high values of distortion $0.336 \lesssim D < 0.50$, $R_{\mathrm{RS}}(D)$ can become slightly smaller than $R_{\mathrm{AT}}(D)$ for a very narrow parameter region, $0.499 \lesssim p \leq 0.5$, which indicates the necessity of introducing the replica symmetry breaking (RSB) solutions. This is also supported analytically by the fact that the inequality $R_{\mathrm{AT}}(D) \sim 2.94\times(p-D)^2 \geq R_{\mathrm{RS}}(D) \sim 2.89\times(p-D)^2$ holds for $p = 0.5$ in the vicinity of $D \lesssim p$. Nevertheless, this instability may not be serious in practice, because the area of the region $R_{\mathrm{RS}}(D) < R < R_{\mathrm{AT}}(D)$, where the RS solution becomes unstable, is extremely small, as indicated by Fig. 5 (a).

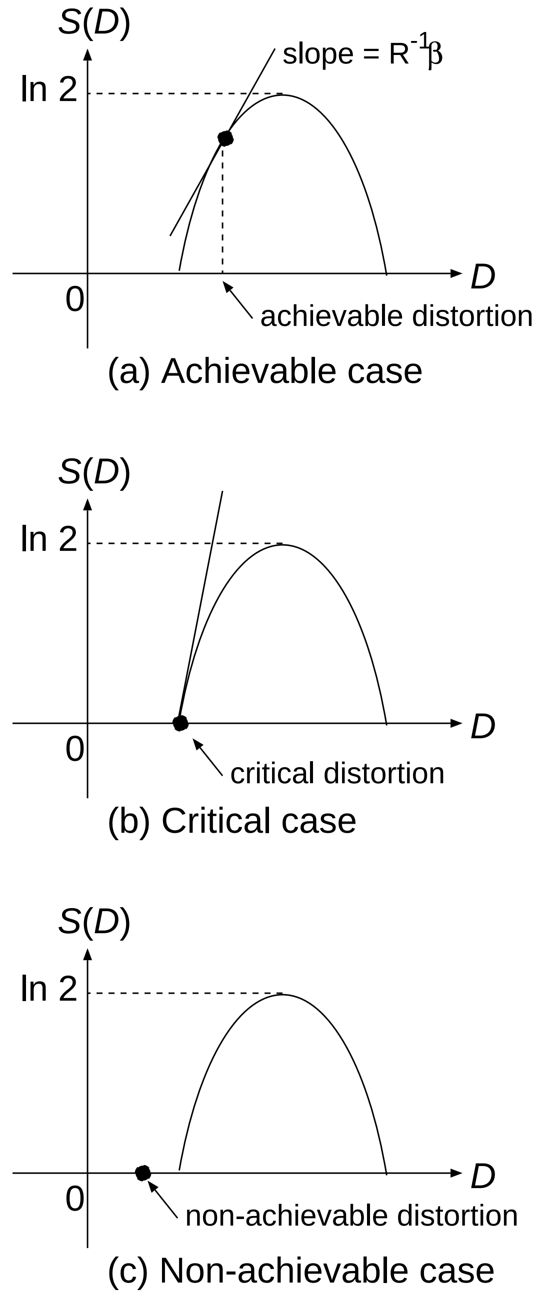


FIG. 3: Schematic profile of the entropy (per bit) $S(D)$. (a): For a modest β, the achieved distortion $D(\beta)$ is such a point that $\partial S(D)/\partial D = R^{-1}\beta$ holds. This is realized by the random sampling from the canonical distribution $P(\boldsymbol{s}|\boldsymbol{y},\boldsymbol{x}^\mu) \sim \exp[-\beta d(\boldsymbol{y},\tilde{\boldsymbol{y}}(\boldsymbol{s}))]$. (b): At a critical inverse temperature $\beta = \beta_c$, the entropy for $D(\beta_c)$ which is the minimum distortion vanishes to zero. (c): It is impossible to achieve any distortion which is smaller than $D(\beta_c)$ as $S(D) = 0$ for $D < D(\beta_c)$.

The other scenario is the coexistence of an RSB solution that is thermodynamically dominant while the RS solution is locally stable. In order to examine this possibility, we solved the saddle point problem assuming the one-step RSB (1RSB) ansatz in several cases for which the RS solution is locally stable. However, no 1RSB solution was discovered for $R \geq R_{\mathrm{RS}}(D)$. Therefore, we concluded that this scenario need not be taken into account in the current system.

These insubstantial roles of RSB may seem somewhat surprising since significant RSB effects above the storage capacity have been reported in the research of perceptrons with continuous couplings [19, 21]. However, this may be explained by the fact that, in most cases, RSB solutions for Ising couplings can be expressed by the RS solutions adjusting temperature appropriately, even if non-monotonic transfer functions are used [17, 22].

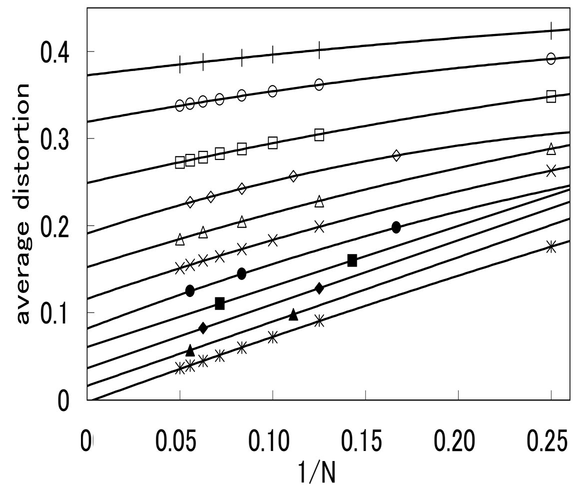
V. NUMERICAL VALIDATION

Although the analysis in the previous section theoretically indicates that the proposed scheme is likely to exhibit a good compression performance, it is still important to confirm it by experiments. Therefore, we have performed numerical simulations implementing the proposed scheme in systems of finite size.

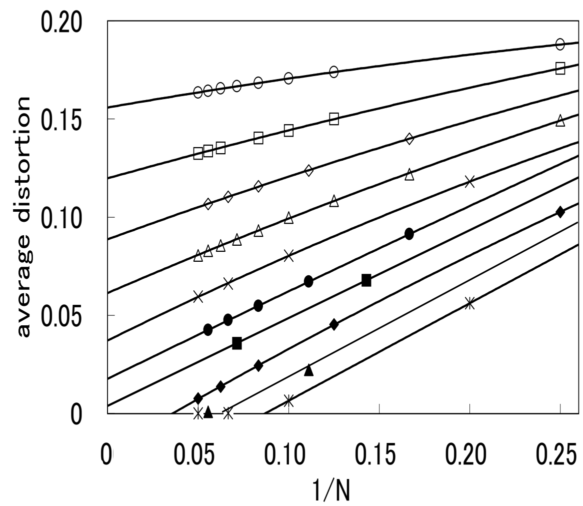
In these experiments, an exhaustive search was performed in order to minimize the distortion $d(\boldsymbol{y}, \tilde{\boldsymbol{y}}(\boldsymbol{s}))$ so as to compress a given message $\boldsymbol{y}$ into $\boldsymbol{s}$, which implies that implementing the current scheme in a large system is difficult. Therefore, validation was performed by extrapolating the numerically obtained data, changing the system size from $N = 4$ to $N = 20$.

Figure 4 shows the average distortions obtained from $5000 \sim 10000$ experiments for (a) unbiased ($p = 0.5$) and (b) biased ($p = 0.2$) messages, varying the system size N and the compression rate $R(= 0.05 \sim 1.0)$. For each R, the threshold parameter k is tuned to the value determined using Eqs. (21), (22) and the rate-distortion function $R = R(D)$ in order to optimize the performance.

These data indicate that the finite size effect is relatively large in the present system, which is similar to the case of the storage capacity problem [26], and do not necessarily seem consistent with the theoretical prediction obtained in the previous section. However, the extrapolated values obtained from the quadratic fitting with respect to $1/N$ are highly consistent with curves of the rate-distortion function (Fig. 5 (a) and (b)), including one point in the region where the AT stability is broken (inset of Fig. 5(a)), which strongly supports the validity and efficacy of our calculation based on the RS ansatz.



(a) $p = 0.5$

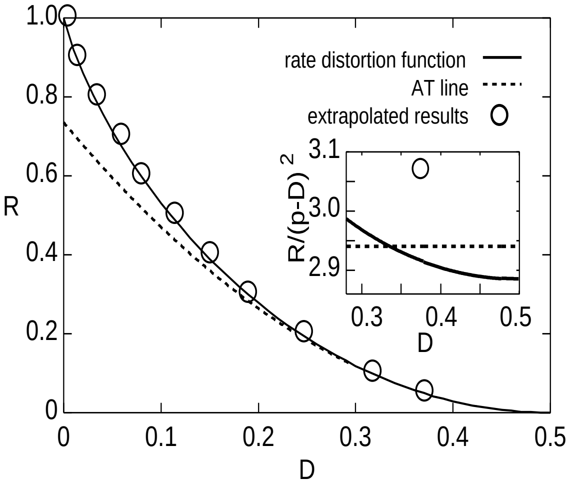


(b) $p = 0.2$

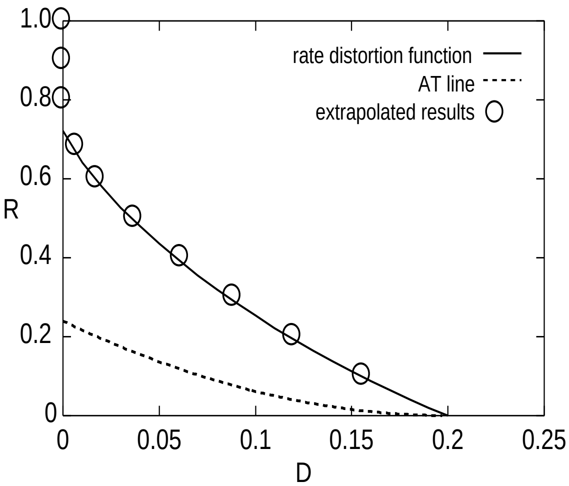
FIG. 4: The averages of the achieved distortions are plotted as functions of $1/N$ for (a) $p = 0.5$ (unbiased) and (b) $p = 0.2$ (biased) messages changing the compression rate R. The plots are obtained from $5000 \sim 10000$ experiments for $N = 4 \sim 20$, minimizing the distortion $d(\boldsymbol{y}, \tilde{\boldsymbol{y}}(\boldsymbol{s}))$ by means of exhaustive search. Each set of plots corresponds to $R = 0.05$ ($p = 0.5$ only), 0.1, 0.2, ..., 1.0, from the top.

VI. SUMMARY AND DISCUSSION

We have investigated a lossy data compression scheme of uniformly biased Boolean messages employing a perceptron of which the transfer function is non-monotonic. Designing the transfer function based on the properties required for good compression codes, we have constructed a scheme that saturates the rate-distortion function that represents the optimal performance in the framework of



(a) $p = 0.5$



(b) $p = 0.2$

FIG. 5: The limits of the achievable distortion expected for $N \to \infty$ are plotted versus the code rate R for (a) $p = 0.5$ (unbiased) and (b) $p = 0.2$ (biased) messages. The plots are obtained by extrapolating the numerically obtained data for systems of $N = 4 \sim 20$ shown in Fig. 4. The full and dashed curves represent the rate-distortion functions and the AT lines, respectively. Although the AT stability is broken for $D \gtrsim 0.336$ for $p = 0.5$ (inset of (a)), the numerical data is highly consistent with the RS solution which corresponds to the rate-distortion function.

lossy compression in most cases.

It is known that a non-monotonic single layer perceptron can be regarded as equivalent to certain types of multi-layered networks, as in the case of parity and committee machines. Although tuning the input-output relation in multi-layered networks would be more complicated, employing such devices might be useful in practice because several heuristic algorithms that could be used for encoding in the present context have been proposed and investigated [27, 28].

In real world problems, the redundancy of information sources is not necessarily represented as a uniform bias; but rather is often given as non-trivial correlations among components of a message. Although it is just unfortunate that the direct employment of the current method may not show a good performance in such cases, the locally activated transfer function $f_{\rm LA}(u)$ that we have introduced herein could serve as a useful building-block to be used in conjunction with a set of connection vectors $\boldsymbol{x}^{\mu=1,2,\ldots,M}$ that are appropriately correlated for approximately expressing the given information source, because by using this function, we can easily control the input-output relation suppressing the bias of the compressed message to zero, no matter how the redundancy is represented.

Finally, although we have confirmed that our method exhibits a good performance when executed optimally in a large system, the computational cost for compressing a message may render the proposed method impractical. One promising approach for resolving this difficulty is to employ efficient approximation algorithms such as various methods of the Monte Carlo sampling [29] and of the mean field approximation [30]. Another possibility is to reduce the finite size effect by further tuning the profile of the transfer function. Investigation of these subjects is currently under way.

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[1] N. Sourlas, Nature (London) **339**, 693 (1989).
[2] D. J. C. MacKay and R. M. Neal, Electron. Lett. **33**, 457 (1997).
[3] H. Nishimori and K. Y. Michael Wong, Phys. Rev. E **60**, 132 (1999).
[4] Y. Kabashima and D. Saad, Europhys. Lett. **45**, 97 (1999).
[5] Y. Kabashima, T. Murayama, and D. Saad, Phys. Rev. Lett. **84**, 1355 (2000).
[6] T. Richardson and R. Urbanke, IEEE Trans. Inf. Theory **47**, 599 (2001).
[7] S. Aji, H. Jin, A. Khandekar, D. J. C. MacKay, and R. J. McEliece, to appear in the Proc. of the IMA 1999 Summer Program: Codes, Systems and Graphical Models, Aug 1999.
[8] T. M. Cover and J. A. Thomas, *Elements of Information*

Theory (John Wiley & Sons, 1991).
[9] C. E. Shannon, Bell Syst. Tech. J. **27**, 379 (1948); **27**, 623 (1948).
[10] C. E. Shannon, IRE National Convention Record, Part 4 142 (1959).
[11] F. Jelinek, *Probabilistic Information Theory* (McGraw-Hill, 1968).
[12] E. Yang, Z. Zhang, and T. Berger, IEEE Trans. Inf. Theory **43**, 1465 (1997).
[13] Y. Matsunaga and H. Yamamoto, Proceedings of 2002 IEEE International Symposium on Information Theory, 461 (2002).
[14] T. Murayama and M. Okada, in preparation.
[15] T. Murayama, J. Phys. A: Math. and Gen. **35**, L95 (2002).
[16] E. Gardner and B. Derrida, J. Phys. A: Math. and Gen. **21**, 271 (1988).
[17] W. Krauth and M. Mézard, J. Phys. (France) **50**, 3057 (1989).
[18] T. Hosaka, *Statistical Mechanics of Lossy Data Compression*, Master's Thesis, Department of Computational Intelligence and Systems Science, Tokyo Institute of Technology (2002).
[19] R. Monasson and D. O'kane, Europhys. Lett. **27**, 85 (1994).
[20] G. J. Bex, R. Serneels, and C. Van den Broeck, Phys. Rev. E **51**, 6309 (1995).
[21] J. van Mourik, K. Y. M. Wong, and D. Bollé, J. Phys. A: Math. and Gen. **33**, L53 (2000).
[22] J. Inoue and D.M. Carlucchi, cond-mat/9806037 (1998).
[23] M. Mézard, G. Parisi, and M. A. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1987).
[24] H. Nishimori, *Statistical Physics of Spin Glasses and Information Processing* (Oxford University Press, 2001).
[25] J. R. L. de Almeida and D. J. Thouless, J. Phys. A: Math. and Gen. **11**, 983 (1977).
[26] W. Krauth and M. Opper, J. Phys. A: Math. and Gen. **22**, L519 (1989).
[27] G.J. Mitchison and R.M. Durbin, Biol. Sybern. **60**, 345 (1989).
[28] K. Nokura, Phys. Rev. E **49**, 5812 (1994).
[29] R. M. Neal, *Bayesian Learning for Neural Networks* (Springer Verlag, 1996).
[30] M. Opper and D. Saad (Eds.), *Advanced Mean Field Methods* (MIT Press, 2001).

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Abstract